Applied Mathematics Letters 25 (2012) 1725–1728

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Computing well diagrams for vector fields on $\mathbb{R}^n$

Frédéric Chazal [a], Primoz Skraba [b], Amit Patel [a,*]

[a] Geometrica, INRIA-Saclay, France
[b] Jozef Stefan Institute, Slovenia

ARTICLE INFO

Article history:
Received 3 September 2011
Accepted 30 January 2012

Keywords:
Well groups
Persistence
Fixed points
Stability
Algorithms

ABSTRACT

Using topological degree theory, we present a fast algorithm for computing the well diagram, a quantitative property, of a vector field on Euclidean space.

1. Introduction

Persistent homology is a popular tool in the field of topological data analysis. It takes as input a filtration of a space, usually derived from a function $f : \mathbb{X} \to \mathbb{R}$, and outputs a persistence diagram [1]. The newer idea of the well diagram is inspired by the persistence diagram [2]. Given a mapping $f : \mathbb{X} \to \mathbb{Y}$ and a subspace $A \subseteq \mathbb{Y}$, the well diagram encodes the robustness of the homology of $f^{-1}(A)$ to perturbations of the mapping f. The well diagram is a finite representation of a zigzag of well groups [3]. Except for in a few special cases, there is as yet no general method for computing a well group [4,5].

In this work, we focus on computing the well diagram for a vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ on $\mathbb{R}^n$ where $A = \{0\}$. The well diagram is interesting because it is both a quantitative and a stable property of the zeros of a vector field. We show that the rank of a well group is completely determined by the topological degree of an appropriate mapping. This insight leads to a fast algorithm. In Section 2, we define the degree diagram and illustrate a simple algorithm for computing it. In Section 3, we show that the degree diagram is equivalent to a well diagram. The stability of the degree diagram is then implied by the stability of the well diagram. In Section 4, we present the well diagram for three example vector fields on $\mathbb{R}^2$. Unless otherwise stated, all homology groups are computed using integer coefficients.

2. Degree diagrams

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous proper mapping from an n-dimensional Euclidean space to itself. Define a function $f_0 : \mathbb{R}^n \to \mathbb{R}$ as the ℓ_2-norm, $f_0(x) = \|f(x)\|_2$, of the image of each point in $\mathbb{R}^n$. A value $r > 0$ is a *regular value* of f_0 if the *sublevel set* $\mathbb{F}_r = f_0^{-1}[0, r]$ is an n-manifold, possibly with boundary, and, for all sufficiently small $\varepsilon > 0$, $f_0^{-1}[r - \varepsilon, r + \varepsilon]$ retracts to $f_0^{-1}(r)$. If r is not regular, then it is *critical*.

The Tameness Assumption. We assume that the function f_0 has a finite number of critical values and that the inverse $f^{-1}(0) = \mathbb{F}_0$ is a finite collection of points.

* Corresponding author.
E-mail addresses: frederic.chazal@inria.fr (F. Chazal), primoz.skraba@ijs.si (P. Skraba), amit@akpatel.org (A. Patel).

0893-9659/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.aml.2012.01.046

Orientation. The *local homology group* $\mathsf{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$, for a point $x \in \mathbb{R}^n$, is isomorphic to the integers Z. An *orientation* μ_x at x is the choice of a generator $\mu_x \in \mathsf{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$. An orientation at x extends to an orientation μ_y, for any point $y \in \mathbb{R}^n$. This follows from the fact that if $C \subset \mathbb{R}^n$ is a closed path connected set containing both x and y, then there are isomorphisms: $\mathsf{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong \mathsf{H}_n(\mathbb{R}^n, \mathbb{R}^n - C) \cong \mathsf{H}_n(\mathbb{R}^n, \mathbb{R}^n - \{y\})$. An *orientation* of $\mathbb{R}^n$ is the choice of a generator μ_x, for some point $x \in \mathbb{R}^n$. If $C \subset \mathbb{R}^n$ is a compact path connected n-manifold with a tubular neighborhood, then an orientation on $\mathbb{R}^n$ implies the choice of a *generator* $\mu(C)$ of $\mathsf{H}_n(C, \mathrm{Bd}\, C) \cong \mathsf{Z}$.

Degrees. Let r be a regular value of f_0, and let $C \subseteq \mathbb{F}_r$ be a path connected component of the sublevel set. The mapping f restricted to C, written as $f\,|_C : (C, \mathrm{Bd}\, C) \to (B_r, \mathrm{Bd}\, B_r)$, takes C to the closed ball B_r of radius r centered at the origin. The mapping f restricted to the boundary, $\mathrm{Bd}\, C$, of C, written as $f\,|_{\mathrm{Bd}\, C} : \mathrm{Bd}\, C \to S^{n-1}$, takes $\mathrm{Bd}\, C$ to the boundary of B_r, which is the $(n-1)$-sphere S^{n-1}. The continuous mappings $f\,|_C$ and $f\,|_{\mathrm{Bd}\, C}$ induce the homomorphisms $\mathsf{f}_*\,|_C : \mathsf{H}_n(C, \mathrm{Bd}\, C) \to \mathsf{H}_n(B_r, \mathrm{Bd}\, B_r)$ and $\mathsf{f}_*\,|_{\mathrm{Bd}\, C} : \mathsf{H}_{n-1}(\mathrm{Bd}\, C) \to \mathsf{H}_{n-1}(S^{n-1})$. Recall that the *degree* of the mapping $f\,|_C$ is the unique integer $\deg(f\,|_C)$ such that $\mathsf{f}_*\,|_C(\mu(C)) = \deg(f\,|_C) \cdot \mu(B_r)$ [6, p. 134].

The boundary homomorphisms between the chain groups induce the following two homomorphisms on the homology, where the second is an isomorphism: $\partial_C : \mathsf{H}_n(C, \mathrm{Bd}\, C) \to \mathsf{H}_{n-1}(\mathrm{Bd}\, C)$ and $\partial_S : \mathsf{H}_n(B_r, \mathrm{Bd}\, B_r) \to \mathsf{H}_{n-1}(S^{n-1})$. Denote by $\nu(\mathrm{Bd}\, C)$ the class $\partial_C(\mu(C))$ and by $\nu(S^{n-1})$ the class $\partial_S(\mu(B_r))$. The *degree* of $f\,|_{\mathrm{Bd}\, C}$ is the unique integer $\deg(f\,|_{\mathrm{Bd}\, C})$ such that $\mathsf{f}_*\,|_{\mathrm{Bd}\, C}(\nu(\mathrm{Bd}\, C)) = \deg(f\,|_C) \cdot \nu(S^{n-1})$. Note that $\mathrm{Bd}\, C$ may not be connected.

Lemma 1. $\deg(f\,|_C) = \deg(f\,|_{\mathrm{Bd}\, C})$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \mathsf{H}_n(C, \mathrm{Bd}\, C) & \xrightarrow{\partial_C} & \mathsf{H}_{n-1}(\mathrm{Bd}\, C) \\ \downarrow{\mathsf{f}_*\,|_C} & & \downarrow{\mathsf{f}_*\,|_{\mathrm{Bd}\, C}} \\ \mathsf{H}_n(B_r, \mathrm{Bd}\, B_r) & \xrightarrow[\cong]{\partial_S} & \mathsf{H}_{n-1}(S^{n-1}). \end{array}$$

The composition $\mathsf{f}_*\,|_{\mathrm{Bd}\, C} \circ \partial_C$ applied to $\mu(C)$ is $\deg(f\,|_{\mathrm{Bd}\, C}) \cdot \nu(S^{n-1})$. The composition $\mathsf{f}_*\,|_C \circ \partial_S$ applied to $\mu(C)$ is $\deg(f\,|_C) \cdot \nu(S^{n-1})$. The diagram commutes, and therefore the statement of the lemma holds. $\square$

There is a third, and also equivalent, definition of a degree associated with f and C. Let $\{x_1, \ldots, x_k\}$ be the collection of points $\mathbb{F}_0 \cap C$. Choose a regular value r of f_0 smaller than the smallest critical value. Then $\mathbb{F}_r \cap C$ has k path connected components $\{D_1, \ldots, D_k\}$ such that $x_i \in D_i$. The mapping f induces a homomorphism $\mathsf{f}_*\,|_i : \mathsf{H}_n(D_i, D_i - \{x_i\}) \to \mathsf{H}_n(B_r, B_r - \{0\})$, for each $1 \le i \le k$. The *local degree* of f at x_i is the unique integer $\deg(f_i)$ such that $\mathsf{f}_*\,|_i(\mu(D_i)) = \deg(f_i) \cdot \mu_0$.

Lemma 2. $\deg(f\,|_C) = \sum_{i=1}^k \deg(f_i)$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} \mathsf{H}_n(C, \mathrm{Bd}\, C) & \xrightarrow{\mathsf{m}} & \mathsf{H}_n(C, C - \mathbb{F}_0) \\ \downarrow{\mathsf{f}_*\,|_C} & & \downarrow{\mathsf{f}_*\,|_{\mathbb{F}_0}} \\ \mathsf{H}_n(B_r, \mathrm{Bd}\, B_r) & \xrightarrow[\cong]{\mathsf{I}} & \mathsf{H}_n(B_r, B_r - \{0\}). \end{array}$$

Both I and m are induced by inclusion, and I is an isomorphism. The homomorphism $\mathsf{f}_*\,|_{\mathbb{F}_0}$ is induced by the mapping f. The class $\mathsf{m}(\mu(C))$ is the direct sum $\oplus_i\, \mu(D_i)$. We have $(\mathsf{f}_*\,|_{\mathbb{F}_0} \circ \mathsf{m})(\mu(C)) = \sum_{i=1}^k \deg(f_i) \cdot \mu_0$. Furthermore $(\mathsf{I} \circ \mathsf{f}_*\,|_C)(\mu(C)) = \deg(f\,|_C) \cdot \mu_0$. The diagram commutes, and therefore the statement of the lemma holds. $\square$

The degree diagram. A *degree diagram* $\mathrm{DDgm}(f_0)$ of f_0 is a multiset of points taken from the extended interval $[0, \infty]$. If r is a regular value of f_0, then the number of points in $\mathrm{DDgm}(f_0)$ with values greater than r is defined as the number of components $C \subseteq \mathbb{F}_r$ such that $\deg(f\,|_C) \neq 0$. For reasons of stability, which will be explained in the next section, the point 0 belongs to $\mathrm{DDgm}(f_0)$ with infinite multiplicity. By the Tameness Assumption, $\mathrm{DDgm}(f_0)$ has only finitely many positive valued points.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous mapping as shown in Fig. 1. There are four points in $\mathbb{F}_0$. From left to right, the local degrees of the four points are $+1, -1, +1$, and -1. The tree in Fig. 1 shows how the components of the sublevel sets $\mathbb{F}_r$ evolve. By Lemma 2, the local degrees are sufficient to determine the degree of any component C in $\mathbb{F}_r$, for r regular. For all sufficiently small $\varepsilon > 0$, there are four components with nonzero degree in $\mathbb{F}_{r_1-\varepsilon}$ and two components with nonzero degree in $\mathbb{F}_{r_1+\varepsilon}$. Therefore, there are two points in the degree diagram $\mathrm{DDgm}(f_0)$ with value r_1. Similarly, there are two points in $\mathrm{DDgm}(f_0)$ with value r_2. The point 0 in the diagram has infinite multiplicity.

3. Stability

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a second proper mapping, and assume that g_0 satisfies the Tameness Assumption. Define the *distance*, $\mathrm{Dist}(f, g)$, between the two mappings as $\mathrm{Dist}(f, g) = \sup_{x \in \mathbb{R}^n} \|f(x) - g(x)\|_2$. There is a notion of a distance

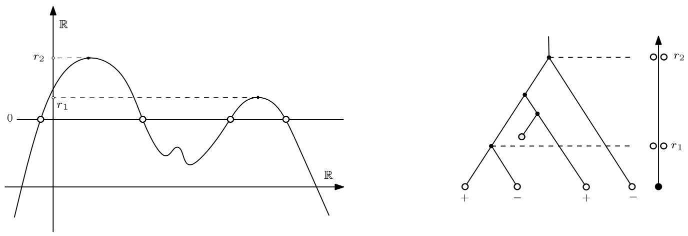

Fig. 1. The graph of a mapping $f : \mathbb{R} \to \mathbb{R}$. The tree encodes the evolution of the components of the sublevel sets $\mathbb{F}_r$ and the degrees of each component at every regular value r. The degree diagram $\mathrm{DDgm}(f_0)$ has four positive valued points.

between the two diagrams $\mathrm{DDgm}(f_0)$ and $\mathrm{DDgm}(g_0)$. Order the positive points in both diagrams and add zeros so that both sequences have the same length. We have $0 \leq u_1 \leq u_2 \leq \cdots \leq u_m$ and $0 \leq v_1 \leq v_2 \leq \cdots \leq v_m$. Define the distance, $\mathrm{Dist}(\mathrm{DDgm}(f_0), \mathrm{DDgm}(g_0))$, between two degree diagrams as $\mathrm{Dist}(\mathrm{DDgm}(f_0), \mathrm{DDgm}(g_0)) = \max_{1 \leq i \leq m} |u_i - v_i|$.

Theorem 1. $\mathrm{Dist}(\mathrm{DDgm}(f_0), \mathrm{DDgm}(g_0)) \leq \mathrm{Dist}(f, g)$.

Theorem 1 says that the degree diagram is stable under perturbations of the mapping f. In the rest of this section, we show that the degree diagram is equivalent to the well diagram as introduced in [2]. The stability of the well diagram implies Theorem 1.

Well diagrams. Let $h : \mathbb{R}^n \to \mathbb{R}^n$ be a mapping such that $\mathrm{Dist}(f, h) < r$. Then $h^{-1}(0) \subseteq \mathbb{F}_r$ induces a homomorphism $\mathsf{h}_* : \mathsf{H}_n(h^{-1}(0); \mathsf{Z}/2\mathsf{Z}) \to \mathsf{H}_n(\mathbb{F}_r; \mathsf{Z}/2\mathsf{Z})$. The well group, $\mathsf{U}(r; \mathsf{Z}/2\mathsf{Z})$, is the subgroup of $\mathsf{H}_n(\mathbb{F}_r; \mathsf{Z}/2\mathsf{Z})$ such that

$$\mathsf{U}_n(r; \mathsf{Z}/2\mathsf{Z}) = \bigcap_{\forall h \text{ s.t.} \mathrm{Dist}(f,h) < r} \mathrm{im}\, \mathsf{h}_*.$$

Assuming that f_0 satisfies the Tameness Assumption, the rank of $\mathsf{U}_n(0; \mathsf{Z}/2\mathsf{Z})$ is the number of fixed points of f. Furthermore, the rank of the well group decreases monotonically with increasing r. The *well diagram*, $\mathrm{WDgm}(f_0)$, is a multiset of points taken from $[0, \infty]$. A point $r \in (0, \infty)$ belongs to $\mathrm{WDgm}(f_0)$ with multiplicity k if the rank of the well group $\mathsf{U}(r)$ changes by k at r. The point ∞ is counted with multiplicity k if for all sufficiently large values of r, the rank of $\mathsf{U}(r)$ is k. The point 0 is counted with infinite multiplicity. The distance between two well diagrams is defined in the same way as the distance between two degree diagrams. In [2], it was shown that $\mathrm{Dist}(\mathrm{WDgm}(f_0), \mathrm{WDgm}(g_0)) \leq \mathrm{Dist}(f, g)$. The following lemma implies Theorem 1.

Lemma 3. *For a regular value r of f_0, the rank of the well group $\mathsf{U}_n(r; \mathsf{Z}/2\mathsf{Z})$ is the number of connected components $C \subseteq \mathbb{F}_r$ such that* $\deg(f|_C) \neq 0$.

Proof. Consider the following diagram for any h such that $\mathrm{Dist}(f, h) \leq r$:

$$\begin{array}{ccc} \mathsf{H}_n(C, \mathrm{Bd}\, C) & \xrightarrow{\mathsf{i}_*} & \mathsf{H}_n(C, C - h^{-1}(0)) \\ \downarrow {\scriptstyle \mathsf{f}_* |_C} & & \downarrow {\scriptstyle \mathsf{h}_* |_0} \\ \mathsf{H}_n(B_r, \mathrm{Bd}\, B_r) & \underset{\cong}{\xrightarrow{\mathsf{j}_*}} & \mathsf{H}_n(B_r, B_r - \{0\}). \end{array}$$

The homomorphisms i_* and j_* are induced by the inclusion of spaces. The mapping $\mathsf{h}_* |_0$ is induced by h. If $\deg(f|_C) \neq 0$, then, by commutativity, $h^{-1}(0) \cap C$ cannot be empty. Therefore the rank of $\mathsf{U}_n(r; \mathsf{Z}/2\mathsf{Z})$ is at least the number of components C such that $\deg(f|_C) \neq 0$.

If $\deg(f|_C) = \deg(f|_{\mathrm{Bd}\, C}) = 0$, then by the Hopf Extension Theorem, there is an *extension* $g : C \to S^{n-1}$ such that g restricted to $\mathrm{Bd}\, C$ equals f restricted to $\mathrm{Bd}\, C$ [7]. Define a mapping $h : \mathbb{R}^n \to \mathbb{R}^n$ as $h(x) = 0.5 \cdot f + 0.5 \cdot g$. By the definition of a regular value, there is no point in the interior of C that maps to the boundary of B_r, and therefore $\mathrm{Dist}(f, h) < r$. Furthermore, the preimage $h^{-1}(0) \cap C$ is empty. In other words, the component C does not contribute to the well group $\mathsf{U}_n(r; \mathsf{Z}/2\mathsf{Z})$. □

4. Experiments

To get a better feeling for the well digram, we compute the well diagram for a random vector field on three subspaces of the plane: a disk, an annulus, and a thickened figure eight. The vector field on the boundary of each domain points inward with a magnitude of 3, and the magnitude of each vector in the interior is at most 3. See Fig. 2 for pictures of each of the three vector fields. A regular value r, the number of components of the sublevel set $\mathbb{F}_r$ with nonzero degree, is the number of points in the diagram with value at least r. In other words, the rank of the well group $\mathsf{U}_2(r; \mathsf{Z}/2\mathsf{Z})$ is the number of points in the diagram with value at least r. We stop growing the sublevel set at radius 3. In Fig. 2(a) and (e), the rank of the well group $\mathsf{U}_2(3; \mathsf{Z}/2\mathsf{Z})$ is 1, a hint of the Euler characteristic of $\mathbb{F}_3$.

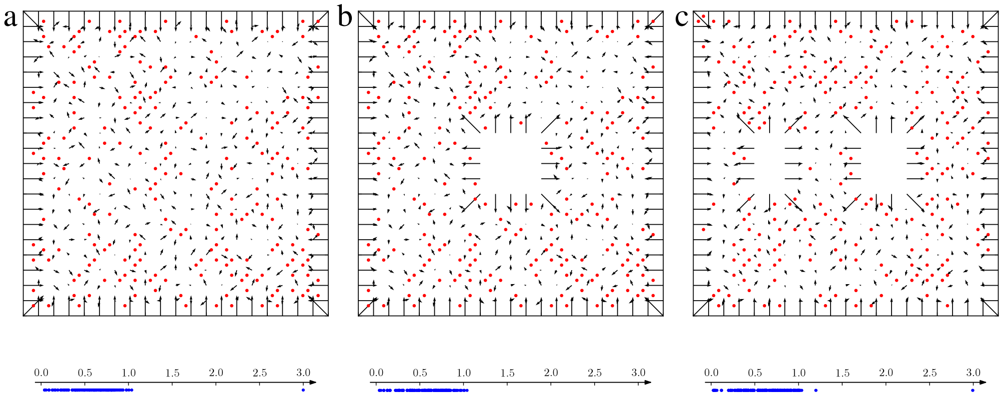

Fig. 2. Vector fields on (a) the disk, (b) the annulus, and (c) the thickened figure eight along with their well diagrams. A red point indicates a fixed point. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

5. Conclusion

We have shown that the seemingly intractable definition of the well group is, for the case of vector fields on $\mathbb{R}^n$, easily understood using degree theory. This observation leads to a fast construction of the well diagram, a new, stable, and quantitative property of a vector field.

Acknowledgments

All three authors would like to thank GIGA project ANR-09-BLAN-0331-01 and Digiteo project C3TTA. Primoz Skraba would like to thank PlanetData FP7-257641, PASCAL2 FP7-216886, X-Like FP7-288342.

References

[1] H. Edelsbrunner, J. Harer, Persistence homology — a survey, Contemporary Mathematics 453 (2008).
[2] H. Edelsbrunner, D. Morozov, A. Patel, Quantifying transversality by measuring the robustness of intersections, Foundations of Computational Mathematics 11 (3) (2011) 345–361.
[3] G. Carlsson, V. de Silva, Zigzag persistence, Foundations of Computational Mathematics 10 (2010) 367–405.
[4] P. Bendich, H. Edelsbrunner, D. Morozov, A. Patel, The robustness of level sets, in: M. de Berg, U. Meyer (Eds.), Algorithms ESA 2010, in: Lecture Notes in Computer Science, vol. 6346, Springer, Berlin, Heidelberg, 2010, pp. 1–10.
[5] H. Edelsbrunner, D. Morozov, A. Patel, The stability of the apparent contour of an orientable 2-manifold, in: V. Pascucci, J. Tierny (Eds.), Topological Methods in Data Analysis and Visualization: Theory, Algorithms, and Applications, Springer-Verlag, 2010.
[6] A. Hatcher, Algebraic Topology, Cambridge University Press, 2002.
[7] V. Guillemin, A. Pollack, Differential Topology, Prentice-Hall Inc., Englewood Cliffs, NJ, 1974.